

Heavy tails and one-dimensional localization.

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Abstract

We address the fundamental questions concerning the operator

$$H^{\theta_0}\psi(x) = -\psi''(x) + V(x, \omega)\psi(x), \quad \psi(0) \cos \theta_0 - \psi'(0) \sin \theta_0 = 0.$$

where the random potential V has a variety of forms. In one example, it is composed of width one bumps of random heights where the square root of the heights are in the domain of attraction of a stable law with index $\alpha \in (0, 1)$ or in another it is composed of width one bumps of height one where the distance between bumps is in the domain of attraction of a stable law with index $\alpha \in (0, 1)$. We consider the existence of Lyapunov exponents, integrated density of states and the nature of the spectrum of the operator.

Keywords: random Schrödinger operator, Lyapunov exponent, rotation number, integrated density of states.

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1 Introduction

In this paper we address a question posed several years ago by G. Zaslowski: what is the effect of heavy tails of one-dimensional random potentials on the standard objects of localization theory: Lyapunov exponents, density of states, statistics of eigenvalues, etc. ? Professor G. Zaslowski always expressed a special interest in the models of chaos containing strong fluctuations, e.g. Lévy flights. We'll consider several models of potentials constructed by the use of *iid* random variables which belong to the domain of attraction of the stable distribution with parameter $\alpha < 1$. In order to put our

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results in context, we'll recall the "regular theory" as presented in Carmona-Lacroix [7] or Figotin-Pastur [10]. Consider the one-dimensional Schrödinger operator on the half line with boundary condition:

$$H^{\theta_0}\psi(x) = -\psi''(x) + V(x, \omega)\psi(x), \psi(0) \cos \theta_0 - \psi'(0) \sin \theta_0 = 0. \quad (1)$$

where for each $x \in [0, \infty)$, $V(x, \cdot)$ is a random variable on a basic probability space (Ω, \mathcal{F}, P) and $\theta_0 \in [0, \pi]$ is fixed. Our potentials $V(x, \omega)$ will be piecewise constant, these are the so-called Krönig-Penny type potentials.

In place of the energy parameter λ , we'll sometimes work with the frequency $\mathbf{k} = \sqrt{|\lambda|}$. Denote the solution of $H^{\theta_0}\psi = \mathbf{k}^2\psi$ by $\psi_{\mathbf{k}}$. For the solution $\psi_{\mathbf{k}}$ of this equation introduce the phase, $\theta_{\mathbf{k}}$, and magnitude, $r_{\mathbf{k}}$, by the Prüfer formulas:

$$\begin{aligned} \psi_{\mathbf{k}}(x) &= r_{\mathbf{k}}(x) \sin \theta_{\mathbf{k}}(x) \\ \psi'_{\mathbf{k}}(x) &= \mathbf{k} r_{\mathbf{k}}(x) \cos \theta_{\mathbf{k}}(x). \end{aligned} \quad (2)$$

Then the equation

$$-\psi''_{\mathbf{k}}(x) + V(x, \omega)\psi_{\mathbf{k}}(x) = \mathbf{k}^2\psi_{\mathbf{k}}(x) \quad (3)$$

can be written, using

$$Y_{\mathbf{k}}(x) = \begin{pmatrix} \psi_{\mathbf{k}}(x) \\ \mathbf{k}^{-1}\psi'_{\mathbf{k}}(x) \end{pmatrix}$$

and

$$A_{\mathbf{k}}(x) = \begin{pmatrix} 0 & \mathbf{k} \\ \frac{V(x) - \mathbf{k}^2}{\mathbf{k}} & 0 \end{pmatrix}$$

in the form $Y'_{\mathbf{k}}(x) = A_{\mathbf{k}}(x)Y_{\mathbf{k}}(x)$. Setting

$$\Theta_{\mathbf{k}}(x) = Y_{\mathbf{k}}(x) \|Y_{\mathbf{k}}(x)\|^{-1}$$

and noting

$$\frac{1}{2}(r_{\mathbf{k}}^2(x))' = \langle \Theta_{\mathbf{k}}(x), A_{\mathbf{k}}(x)\Theta_{\mathbf{k}}(x) \rangle r_{\mathbf{k}}(x)$$

and combining this with (1), one easily derives the standard Ricatti equations for the phase and magnitude:

$$\begin{aligned} \theta'_{\mathbf{k}}(x) &= \mathbf{k} - \frac{1}{\mathbf{k}}V(x, \omega) \sin^2 \theta_{\mathbf{k}}(x) \\ r'_{\mathbf{k}}(x) &= \frac{1}{2}r_{\mathbf{k}}(x)V(x, \omega) \sin 2\theta_{\mathbf{k}}(x). \end{aligned} \quad (4)$$

The monodromy operator, which is in $SL(2, \mathbf{R})$, denoted by $\mathcal{M}_{\mathbf{k}}$, transforms the values of the solution as follows: for $a < b$,

$$\mathcal{M}_{\mathbf{k}}([a, b]) \begin{pmatrix} \psi_{\mathbf{k}}(a) \\ \mathbf{k}^{-1}\psi'_{\mathbf{k}}(a) \end{pmatrix} = \begin{pmatrix} \psi_{\mathbf{k}}(b) \\ \mathbf{k}^{-1}\psi'_{\mathbf{k}}(b) \end{pmatrix}. \quad (5)$$

We'll use the notation $\mathcal{M}_{\mathbf{k}}(n) \equiv \mathcal{M}_{\mathbf{k}}([n-1, n])$, $n = 1, 2, \dots$

The standard assumptions on $\mathcal{M}_{\mathbf{k}}$ are

- For all n , $E [\ln^+ ||\mathcal{M}_{\mathbf{k}}(n)||] < \infty$.
- The sequence $\{\mathcal{M}_{\mathbf{k}}(n) : n \geq 1\}$ is either independent or forms a stationary sequence with rapidly decaying correlations.
- The random variables $\mathcal{M}_{\mathbf{k}}(n)$ have a density in $SL(2, \mathbf{R})$ with respect to Haar measure, alternatively, the sequence of phases $\{\theta_{\mathbf{k}}(n) : n \geq 1\}$ defined by

$$(\sin \theta_{\mathbf{k}}(n+1), \cos \theta_{\mathbf{k}}(n+1)) \equiv \Theta_{\mathbf{k}}(n+1) = \frac{\mathcal{M}_{\mathbf{k}}(n)\Theta_{\mathbf{k}}(n)}{||\mathcal{M}_{\mathbf{k}}(n)\Theta_{\mathbf{k}}(n)||}$$

satisfies the Döblin condition. That is the Markov chain $\{\theta_{\mathbf{k}}(n) : n \geq 1\}$ has a "good" transition density $p_{\mathbf{k}}(\theta, \eta)$.

Under these assumptions, there exists a continuous and strictly positive Lyapunov exponent

$$\gamma(\mathbf{k}^2) = \lim_{n \rightarrow \infty} \frac{1}{n} \ln ||\prod_{j=1}^n \mathcal{M}_{\mathbf{k}}(\mathbf{j})||, \text{ a.s.}$$

and an integrated density of states

$$N(\mathbf{k}^2) = \lim_{n \rightarrow \infty} \frac{\theta_{\mathbf{k}}(n)}{\pi n}, \text{ a.s..}$$

The integrated density of states and Lyapunov exponent are related by the Thouless formula, see e.g. [7],

$$\gamma(\mathbf{k}^2) = \gamma_0(\mathbf{k}^2) + \int_{\mathbf{R}} \ln |\mathbf{k}^2 - \mathbf{u}| (\mathbf{N}(\mathbf{d}\mathbf{u}) - \mathbf{N}_0(\mathbf{d}\mathbf{u}))$$

where $\gamma_0(u) = \sqrt{u}$ and $\mathbf{N}_0(\mathbf{u}) = \frac{\sqrt{u}}{\pi}$ are the Lyapunov exponent and integrated density of states for the operator $H_0 = -\frac{d^2}{dx^2}$. Finally, for a.e θ_0 with respect to Lebesgue measure, the spectrum of H^{θ_0} is pure point *a.s.* and the eigenfunction corresponding to the eigenvalue \mathbf{k}^2 is exponentially decreasing at rate $\gamma(\mathbf{k}^2)$ which is referred to as the exponential localization theorem. Typical references for these results would be [4], [8], [13], [19], and an overview of the field is contained in the lecture notes [16].

In contrast to the preceding classical situation, we'll show that for heavy-tailed potentials, of Krönig-Penny type, (meaning the tail of the distribution of the random potential decays slowly) the limits defining either $N(\mathbf{k}^2)$ or $\gamma(\mathbf{k}^2)$ do not exist under the usual normalization. However, after appropriate non-linear normalization, they converge in distribution to non-degenerate random variables. Under appropriate assumptions on the tails, these random variables have a stable distribution. The underlying phenomenon is related to Darling's Theorem, [5], on the contribution of the maximal term to a sum of random variables in the domain of attraction of a stable law with index $\alpha \in (0, 1)$. Namely, if $\{\zeta_n\}_{n \geq 1}$ are iid, nonnegative random

variables with tails given by $P(\zeta_1 > x) = \frac{L(x)}{x^\alpha}$ where L is a slowly varying function, $S_n = \sum_{j=1}^n \zeta_j$ and $\zeta_n^* = \max_{1 \leq j \leq n} \zeta_j$ then the ratio ζ_n^*/S_n has a nondegenerate limiting law. The quantities we examine have the form of S_n with summands in the domain of attraction of a stable law with index $\alpha \in (0, 1)$. Such sums can not be normalized to converge *a.s.* to a nonzero deterministic constant since from time to time a new summand will have the same order of magnitude as the entire sum. This prevents the usual self averaging we see in the classical case of ergodic potentials. It's important to emphasize that our potentials are not ergodic, so the usual theorems that apply to Schrödinger operators with ergodic potentials do not apply in our models. For example, in **Model III** we have a Lyapunov exponent which is identically zero. If the potential were ergodic, this would imply the spectrum is *a.s.* absolutely continuous, see [4], Theorem 4. In our model we find the spectrum **Model III** is *a.s.* pure point in spite of having a vanishing Lyapunov exponent. We now give a description of four models that will be covered in this paper and the interesting effects that they display.

Model I

In the first model, V has the form

$$V(x, \omega) = \sum_{n=0}^{\infty} 1_{[n, n+1)}(x) X_n(\omega) \quad (6)$$

where $\{X_n : n \geq 0\}$ are *iid* random variables with common density p . This density will be assumed to be bounded, continuous and satisfy $p(x) > 0$ for $x > 0$ and vanishing identically for $x \leq 0$. Moreover, we shall assume that $\sqrt{X_n}$ belongs to the domain of attraction of an α -stable law, denoted St_α , where $0 < \alpha < 1$. This assumption is equivalent to requiring that $P(\sqrt{X_n} > x) = \frac{L(x)}{x^\alpha}$, $x \rightarrow \infty$, with L a slowly varying function. In **Model I** we will show that for *a.e.* θ_0 , P *a.s.*, the spectrum of H^{θ_0} is pure-point and the eigenfunctions decay super-exponentially. Also, it turns out that a properly normalized integrated density of states $N(\mathbf{k}^2)$ exists and is continuous in spite of the fact that $E[\ln \|\mathcal{M}_{\mathbf{k}}(n)\|] = \infty$. Perhaps the most interesting result here concerns the Lyapunov exponent. We'll demonstrate that

$$\ln \|\mathcal{M}_{\mathbf{k}}([0, n])\| = \ln \|\mathcal{M}_{\mathbf{k}}(1)\mathcal{M}_{\mathbf{k}}(2) \cdots \mathcal{M}_{\mathbf{k}}(n)\| \sim \sum_{j=1}^n \sqrt{X_j}.$$

As a consequence, we'll have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln \|\mathcal{M}_{\mathbf{k}}([0, n])\| = \infty, \quad P - a.s.,$$

whereas

$$\lim_{n \rightarrow \infty} \frac{1}{n^{1/\alpha}} \ln \|\mathcal{M}_{\mathbf{k}}([0, n])\| \stackrel{\mathcal{L}}{=} \zeta_\alpha,$$

where ζ_α has an St_α distribution, where \mathcal{L} is used to indicate convergence in law. The convergence in distribution can not be improved to *a.s.* since

$$\zeta(n) \equiv \frac{\ln ||\mathcal{M}_{\mathbf{k}}([0, n])||}{n^{1/\alpha}}$$

randomly oscillates as $n \rightarrow \infty$ while its distribution is tending to the St_α law. This can be seen by the decomposition

$$\begin{aligned} \zeta(2^{n+1}) &\sim 2^{-(n+1)/\alpha} \sum_{j=1}^{2^n} \sqrt{X_j} + 2^{-(n+1)/\alpha} \sum_{j=2^{n+1}}^{2^{n+1}} \sqrt{X_j} \\ &\sim 2^{-1/\alpha} (\zeta(2^n) + \tilde{\zeta}(2^n)) \end{aligned} \quad (7)$$

where $\zeta(2^n)$ and $\tilde{\zeta}(2^n)$ are independent and nearly St_α distributed. Thus, $\zeta(2^{n+1}) - \zeta(2^n) = (2^{-1/\alpha} - 1)\zeta(2^n) + 2^{-1/\alpha}\tilde{\zeta}(2^n)$ and so *a.s.* convergence does not hold in this situation. The results are summarized in the following theorem.

Theorem 1.1. *In Model I, for any \mathbf{k}^2 , the standard integrated density of states exists and is given by*

$$N(\mathbf{k}^2) = \lim_{n \rightarrow \infty} \frac{\theta_{\mathbf{k}}(n)}{\pi n}, \quad P - a.s..$$

In addition, the linear scale Lyapunov exponent is infinite, i.e.

$$\gamma(\mathbf{k}^2) = \lim_{n \rightarrow \infty} \frac{1}{n} \ln ||\mathcal{M}_{\mathbf{k}}([0, n])|| = \infty, \quad P - a.s.,$$

whereas in the nonlinear scale, we have convergence in distribution,

$$\gamma_{\text{nl}}(\mathbf{k}^2) = \lim_{n \rightarrow \infty} \frac{1}{n^{1/\alpha}} \ln ||\mathcal{M}_{\mathbf{k}}([0, n])|| \stackrel{\mathcal{L}}{=} \zeta_\alpha,$$

where ζ_α has an St_α distribution.

For a.e. $\theta_0 \in [0, \pi]$, the operator H^{θ_0} has pure point spectrum a.s.. In addition, the eigenfunctions satisfy

$$\lim_{x \rightarrow \infty} \frac{1}{2x^{1/\alpha}} \ln \left(\psi_{\mathbf{k}}(x)^2 + \frac{1}{\mathbf{k}^2} \psi'_{\mathbf{k}}(x)^2 \right) \stackrel{\mathcal{L}}{=} -\xi_\alpha.$$

Model II

This model is the same as **Model I** except now the potential is non-positive. We take

$$V(x, \omega) = - \sum_{n=0}^{\infty} 1_{[n, n+1)}(x) X_n(\omega) \quad (8)$$

with the same conditions on the distribution of X_n as in **Model I**, namely, the random variables X_n have common density p which is bounded, continuous and satisfies $p(x) > 0$ for $x > 0$, $p(x) \equiv 0$ for $x \leq 0$ and $P(\sqrt{X_n} > x) = \frac{L(x)}{x^\alpha}$, where L is a slowly varying function. The corresponding operator is essentially self-adjoint without any assumptions on the tails of X_n . This last fact can't be proven by appealing to Weyl's criterion (which requires the condition $V(x) \geq -c_0 - c_1 x^2$, $c_0 > 0, c_1 > 0$ as can be found in [14].) However, it does follow from a Theorem of Hartman which is stated below. In this model we'll switch back to λ instead of \mathbf{k}^2 since there is spectrum on both sides of the origin in this case. The Lyapunov exponent $\gamma(\lambda)$ is positive and continuous (at least in the situation of **Model I** when the tails aren't too heavy.) However, in **Model II** the density of states $N(\lambda)$ demonstrates unusual behavior:

Theorem 1.2. *In Model II, for any λ ,*

$$\theta_\lambda(n) = \sum_{j=0, X_j \geq -\lambda}^n \sqrt{X_j + \lambda} + O(n), \quad P - a.s.. \quad (9)$$

Consequently,

$$\lim_{n \rightarrow \infty} \frac{\theta_\lambda(n)}{\pi n} = \infty, \quad P - a.s.,$$

but

$$\lim_{n \rightarrow \infty} \frac{\theta_\lambda(n)}{\pi n^{1/\alpha}} \stackrel{\mathcal{L}}{=} \zeta_\alpha,$$

where ζ_α has an St_α distribution. Finally,

$$\gamma(\lambda) = \lim_{n \rightarrow \infty} \frac{1}{n} \ln ||\mathcal{M}_\lambda([0, n])||, \quad P \text{ a.s.} \quad (10)$$

and $\gamma(\lambda) > 0$. Consequently, H^{θ_0} has pure point spectrum for a.e. θ_0 . In addition, the eigenfunctions satisfy

$$\lim_{n \rightarrow \infty} \frac{1}{x} \ln r_\lambda(x) = \lim_{x \rightarrow \infty} \frac{1}{2x} \ln \left(\psi_\lambda(x)^2 + \frac{1}{\lambda} \psi'_\lambda(x)^2 \right) = -\gamma(\lambda).$$

Model III

For this model, $\{Y_n : n \geq 1\}$ are *iid* random variables with common density p which is bounded, satisfies $p(x) > 0$ for $x > 0$ and $P(Y_1 > x) = \frac{L(x)}{x^\alpha}$, $\alpha \in (0, 1)$, for some slowly varying function L . Set

$$L_n = S_n + n = \sum_{j=1}^n Y_j + n \quad (11)$$

and define

$$V(x, \omega) = \sum_{j=1}^{\infty} \mathbf{1}_{[L_j(\omega)-1, L_j(\omega)]}(x).$$

This potential is a system of bumps of height 1 and width 1 with the j^{th} and $(j+1)^{st}$ bumps being separated by the random distance Y_{j+1} . This could be generalized easily to bumps of height h and width $\delta > 0$, with no complications. Under the assumption that $\int_x^\infty p(y)dy = \frac{L(x)}{x^\alpha}$, with L a slowly varying function, the distance between bumps exhibits strong fluctuations. Observe that the right edge of the n^{th} bump has distance $S_n + n \equiv L_n$ from the origin. Under our assumption on the tail behavior of $p(x)$, it follows that

$$\frac{L_n}{n^{1/\alpha}} \xrightarrow{\mathcal{L}} \zeta_\alpha,$$

where, as before, ζ_α has a St_α distribution. For this model we have the following results.

Theorem 1.3. *In Model III,*

$$\theta_{\mathbf{k}}(L_n) = \mathbf{k} \sum_{j=1}^n Y_j + O(n).$$

The integrated density of states exists but,

$$N(\mathbf{k}^2) = \lim_{\mathbf{x} \rightarrow \infty} \frac{\theta_{\mathbf{k}}(\mathbf{x})}{\pi \mathbf{x}} = \infty, \text{ a.s.},$$

while

$$\lim_{x \rightarrow \infty} \frac{\theta_{\mathbf{k}}(x)}{\pi x^{1/\alpha}} \stackrel{\mathcal{L}}{=} \mathbf{k} \zeta_\alpha.$$

The Lyapunov exponent in the linear scale is

$$\gamma(\mathbf{k}^2) = \lim_{\mathbf{n} \rightarrow \infty} \frac{1}{\mathbf{L}_{\mathbf{n}}} \ln ||\mathcal{M}_{\mathbf{k}}([0, \mathbf{L}_{\mathbf{n}}])|| = \mathbf{0} \quad (12)$$

while there exists a Lyapunov exponent in the 'non-linear' scale

$$\gamma(\mathbf{k}^2)_{\text{nl}} = \lim_{\mathbf{n} \rightarrow \infty} \frac{1}{\mathbf{n}} \ln ||\mathcal{M}_{\mathbf{k}}([0, \mathbf{L}_{\mathbf{n}}])|| > \mathbf{0}$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{L_n^\alpha} \ln ||\mathcal{M}_{\mathbf{k}}([0, L_n])|| \stackrel{\mathcal{L}}{=} \frac{\gamma(\mathbf{k}^2)_{\text{nl}}}{\zeta_\alpha^\alpha}.$$

For a.e. θ_0 , H^{θ_0} has pure point spectrum a.s.. Moreover, if $\mathbf{k}^2 \in \Sigma(H^{\theta_0})$ then the corresponding eigenfunction satisfies

$$\lim_{x \rightarrow \infty} \frac{1}{2x^\alpha} \ln (\psi_{\mathbf{k}}^2(x) + \psi_{\mathbf{k}}'^2(x)) \stackrel{\mathcal{L}}{=} -\frac{\gamma(\mathbf{k}^2)_{\text{nl}}}{\zeta_\alpha^\alpha}.$$

Model IV

The final model incorporates the strong fluctuations in the bump size from **Model I** and **Model II** as well as the strong fluctuation in the gap size as in **Model III**. The results for this model are similar to those for the first three models and so we won't state the results here as a Theorem but rather confine ourselves to a description of what can be proven. In this model we will denote the bump sizes by the *iid* random variables $\{X_n\}$ and the gap sizes by $\{Y_n\}$ again assumed to be *iid*. As before we will assume that $\{\sqrt{X_n}\}$ and $\{Y_n\}$ have densities $p_i(x)$, $i = 1, 2$ respectively, which are bounded, positive for $x > 0$ and have tail behavior $\int_x^\infty p_i(y)dy = \frac{L_i(x)}{x^{\alpha_i}}$, $i = 1, 2$, $\alpha_1, \alpha_2 \in (0, 1)$ where the L_i are slowly varying as above so that the random variable $\{\sqrt{X_n}\}$ and $\{Y_n\}$ are in the domain of attraction of stable laws,

$$\lim_{n \rightarrow \infty} \frac{1}{n^{1/\alpha_1}} \sum_{j=1}^n \sqrt{X_j} \stackrel{\mathcal{L}}{=} \zeta_{\alpha_1}$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{n^{1/\alpha_2}} \sum_{j=1}^n Y_j \stackrel{\mathcal{L}}{=} \zeta_{\alpha_2}$$

where ζ_{α_1} and ζ_{α_2} have respectively St_{α_1} and St_{α_2} distributions. The explicit form of the potential here is

$$V(x) = \sum_{n=1}^{\infty} (-1)^{\epsilon_n} X_n 1_{[L_n-1, L_n]}(x), \quad (13)$$

where $L_n = \sum_{j=1}^n Y_j + n$ and $\{\epsilon_n : n \geq 1\}$ is an iid sequence of Bernoulli random variables $P(\epsilon_n = 1) = P(\epsilon_n = 0) = \frac{1}{2}$ and they are independent of the sequences $\{X_n : n \geq 1\}$ and $\{Y_n : n \geq 1\}$. In this model, again switching back to λ instead of \mathbf{k}^2 due to the appearance of spectrum to the right of the origin. A typical result is that

$$\ln ||\mathcal{M}_\lambda([0, L_n])|| \sim \sum_{j=1}^n \sqrt{X_j}$$

and

$$\begin{aligned} \frac{\ln ||\mathcal{M}_\lambda([0, L_n])||}{L_n} &\sim \frac{\sum_{j=1}^n \sqrt{X_j}}{\sum_{j=1}^n Y_j} \\ &= \frac{n^{-1/\alpha} \sum_{j=1}^n \sqrt{X_j}}{n^{-1/\alpha} \sum_{j=1}^n Y_j} \\ &\xrightarrow{\mathcal{L}} \frac{\zeta_{\alpha_1}}{\zeta_{\alpha_2}}, \end{aligned} \quad (14)$$

where ζ_{α_1} and ζ_{α_2} are independent with St_{α_1} , St_{α_2} distributions, respectively. The rotation number takes the form

$$\theta_\lambda(n) = \lambda \sum_{i=1}^n Y_i + \sum_{i=1, \epsilon_i=-1}^n \sqrt{X_i + \lambda} + O(n).$$

The first sum on the right hand side is the accumulated rotation across the gaps as in **Model III**. The second term is result of the rotation across the negative bumps as in **Model II**. The $O(n)$ term is the effect of the positive bumps as in **Model I**. Thus, there is a nonlinear version of the integrated density of states which depends on which index α_1 or α_2 is smaller. Setting $\alpha = \alpha_1 \wedge \alpha_2$, we have the limit

$$N_{nl}(\lambda) = \lim_{n \rightarrow \infty} \frac{\theta_\lambda(n)}{\pi n^{1/\alpha}} \stackrel{\mathcal{L}}{=} \zeta_\alpha,$$

where as usual, ζ is a random variable with a St_α distribution. These follow in a manner similar to the results for **Models I, II, III** so we omit the proofs in this case.

As mentioned above, in place of the energy parameter λ , in **Model I** and **Model III** we'll work with the frequency $\mathbf{k} = \sqrt{|\lambda|}$. For the solution ψ_λ of the boundary value problem

$$H^{\theta_0} \psi_{\mathbf{k}} = \mathbf{k}^2 \psi_{\mathbf{k}}$$

the phase, $\theta_{\mathbf{k}}$, and magnitude, $r_{\mathbf{k}}$, are given by the Prüfer formulas, see (2), and satisfy the standard Ricatti equations given at (4). Recall that we denote by $\mathcal{M}_\lambda([a, b])$ the propagator of the system whose action is given in (5).

In the case of **Model I**, the propagator is the product of random matrices,

$$\mathcal{M}_{\mathbf{k}}([0, n]) = \mathcal{A}_{\mathbf{k}}(X_n) \cdots \mathcal{A}_{\mathbf{k}}(X_2) \mathcal{A}_{\mathbf{k}}(X_1),$$

where we have slightly changed the notation used in the introduction for the matrices in this product. The form of these matrices depends on whether $X_l < \mathbf{k}$ or $X_l \geq \mathbf{k}$. In the former case,

$$\mathcal{A}_{\mathbf{k}}(X_l) = \begin{pmatrix} \cos \sqrt{\mathbf{k}^2 - X_l} & \frac{\mathbf{k}}{\sqrt{\mathbf{k}^2 - X_l}} \sin \sqrt{\mathbf{k}^2 - X_l} \\ -\frac{\sqrt{\mathbf{k}^2 - X_l}}{\mathbf{k}} \sin \sqrt{\mathbf{k}^2 - X_l} & \cos \sqrt{\mathbf{k}^2 - X_l} \end{pmatrix}, \quad (15)$$

whereas in the latter,

$$\mathcal{A}_{\mathbf{k}}(X_l) = \begin{pmatrix} \cosh \sqrt{X_l - \mathbf{k}^2} & \frac{\mathbf{k}}{\sqrt{X_l - \mathbf{k}^2}} \sinh \sqrt{X_l - \mathbf{k}^2} \\ \frac{\sqrt{X_l - \mathbf{k}^2}}{\mathbf{k}} \sinh \sqrt{X_l - \mathbf{k}^2} & \cosh \sqrt{X_l - \mathbf{k}^2} \end{pmatrix}. \quad (16)$$

In the case of **Model II**, again the propagator is the product of random matrices,

$$\mathcal{M}_{\mathbf{k}}([0, n]) = \mathcal{B}_{\mathbf{k}}(X_n) \cdots \mathcal{B}_{\mathbf{k}}(X_2) \mathcal{B}_{\mathbf{k}}(X_1),$$

which are simply

$$\mathcal{B}_\lambda(X_l) = \begin{pmatrix} \cos \sqrt{\lambda + X_l} & \frac{\sqrt{|\lambda|}}{\sqrt{\lambda + X_l}} \sin \sqrt{\lambda + X_l} \\ -\frac{\sqrt{\lambda + X_l}}{\sqrt{|\lambda|}} \sin \sqrt{\lambda + X_l} & \cos \sqrt{\lambda + X_l} \end{pmatrix}, \quad (17)$$

for $X_l + \lambda \geq 0$ with a similar matrix using hyperbolic trig functions when $X_l + \lambda < 0$ as in **Model I**, namely

$$\mathcal{A}_\lambda(X_l) = \begin{pmatrix} \cosh \sqrt{-(\lambda + X_l)} & \frac{|\lambda|}{\sqrt{-(\lambda + X_l)}} \sinh \sqrt{-(\lambda + X_l)} \\ \frac{\sqrt{-(\lambda + X_l)}}{|\lambda|} \sinh \sqrt{-(\lambda + X_l)} & \cosh \sqrt{-(\lambda + X_l)} \end{pmatrix}. \quad (18)$$

For **Model III**, the propagator is more easily expressed at the times $L_n = Y_1 + \dots + Y_n + n = S_n + n$ marking the end of the n^{th} gap, then it is the product

$$\mathcal{M}_{\mathbf{k}}([0, L_n]) = \mathcal{C}_{\mathbf{k}}(Y_n) \cdots \mathcal{C}_{\mathbf{k}}(Y_2) \mathcal{C}_{\mathbf{k}}(Y_1),$$

where $\mathcal{C}_{\mathbf{k}}(Y_l)$ factors into the product of the transfer matrix between the bumps multiplied by the transfer matrix across the following bump,

$$\mathcal{C}_{\mathbf{k}}(Y_l) = \tilde{\mathcal{C}}_{\mathbf{k}}(Y_l) \hat{\mathcal{C}}_{\mathbf{k}}(Y_l). \quad (19)$$

The monodromy operator between the bumps is

$$\hat{\mathcal{C}}_{\mathbf{k}}(Y_l) = \begin{pmatrix} \cos \mathbf{k} Y_l & \sin \mathbf{k} Y_l \\ -\sin \mathbf{k} Y_l & \cos \mathbf{k} Y_l \end{pmatrix}, \quad (20)$$

while the form of the operator across the bumps, $\tilde{\mathcal{C}}_{\mathbf{k}}(Y_l)$, is deterministic and its form depends on whether $1 < \mathbf{k}$ or $1 \geq \mathbf{k}$. In the former case,

$$\tilde{\mathcal{C}}_{\mathbf{k}}(Y_l) = \begin{pmatrix} \cos \sqrt{\mathbf{k}^2 - 1} & \frac{\mathbf{k}}{\sqrt{\mathbf{k}^2 - 1}} \sin \sqrt{\mathbf{k}^2 - 1} \\ -\frac{\sqrt{\mathbf{k}^2 - 1}}{\mathbf{k}} \sin \sqrt{\mathbf{k}^2 - 1} & \cos \sqrt{\mathbf{k}^2 - 1} \end{pmatrix}, \quad (21)$$

whereas in the latter,

$$\tilde{\mathcal{C}}_{\mathbf{k}}(Y_l) = \begin{pmatrix} \cosh \sqrt{1 - \mathbf{k}^2} & \frac{\mathbf{k}}{\sqrt{1 - \mathbf{k}^2}} \sinh \sqrt{1 - \mathbf{k}^2} \\ -\frac{\sqrt{1 - \mathbf{k}^2}}{\mathbf{k}} \sinh \sqrt{1 - \mathbf{k}^2} & \cosh \sqrt{1 - \mathbf{k}^2} \end{pmatrix}. \quad (22)$$

The matrices $\{\mathcal{C}_{\mathbf{k}}(Y_l) : l \geq 1\}$ are *iid* elements of $SL(2, \mathbf{R})$.

The monodromy operator for **Model IV** is a mixture of products of matrices of the above form, across a gap of width Y_l a matrix of the form (20) is represented in the product. When a bump of height $(-1)^{\epsilon_l} X_l$ is encountered and $\epsilon_l = 1$, a matrix of the form (15) or (18) appears in the product depending on whether $X_l < \mathbf{k}$ or $X_l \geq \mathbf{k}$. When $\epsilon_l = -1$, then a matrix of the form (17) is entered in the product.

2 Auxilliary Results

For the proof of a.s. localization of **Models I** and **II** we'll use the following classical result.

Theorem 2.1. (*Furstenberg*) *If $\{M_j\}_{j \geq 1}$ are i.i.d. elements of $SL(2, R)$ with $E[\ln ||M_1||] < \infty$, and the corresponding Markov chain*

$$\theta_n = \frac{M_n M_{n-1} \cdots M_1 \theta_0}{||M_n M_{n-1} \cdots M_1 \theta_0||}$$

is ergodic and the distribution of M_1 is not contained in a compact subgroup of $SL(2, R)$ then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln ||M_n M_{n-1} \cdots M_1 \theta_0|| = \gamma > 0, \text{ a.s..} \quad (23)$$

Moreover there is a one dimensional subspace $\mathbf{W} \subset \mathbf{R}^2$ such that for $\theta \in \mathbf{W} \setminus \{\mathbf{0}\}$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln ||M_n M_{n-1} \cdots M_1 \theta|| = -\gamma, \text{ a.s..} \quad (24)$$

An important ingredient in establishing the decay of eigenfunctions is the following result of Sch'nol. A more general version of the following result is Theorem C.4.1 in [18].

Theorem 2.2. (*Sch'nol*) *Suppose V is one of the potentials in **Model I – IV** below and $Hu = \Delta u + Vu = Eu$ with u polynomially bounded. Then $E \in \Sigma(H)$.*

In the classical setting mentioned in the **Introduction**, this has been used in conjunction with the Lyapunov exponents for the transfer matrix to establish that a generalized eigenfunction of polynomial growth must actually decay exponentially. This was the technique used in [6], [13] and [16] for example. In the case of the heavy tailed potentials examined in this work, it establishes super-exponential decay of the eigenfunctions, that is an eigenfunction will satisfy

$$\lim_{x \rightarrow \infty} \frac{1}{x^{1/\alpha}} \ln \sqrt{\psi_{\mathbf{k}}^2(x) + \psi_{\mathbf{k}'}^2(x)} \stackrel{\mathcal{L}}{=} -\zeta_\alpha,$$

where ζ_α is a random variable in the class St_α .

In **Model II** we use the following theorem to establish the essential self-adjointness of H .

Theorem 2.3. (*Hartman's Theorem*)

If one can find a sequence of intervals $\Delta_n = [x_n, x_n + \delta]$ with $\delta > 0$ fixed and $x_n \rightarrow \infty$ and a constant $c_0 > -\infty$ such that $V(x) \geq c_0$ on Δ_n , then the operator H is essentially self-adjoint.

This applies to **Model II**,

Corollary 2.1. *The operators H^{θ_0} in **Models II** and **IV** are essentially self-adjoint.*

Proof. Just use Borel-Cantelli, $\sum_{n=1}^{\infty} P(-X_n > c_0) = \infty$ so there is a.s. an infinite subsequence X_{n_k} such that $-X_{n_k} > c_0$ for all k . Then taking $x_k = X_{n_k}$ and $\delta = 1$ the conditions of Hartman's Theorem are satisfied. Thus the operators defined in **Models II** and **IV** are essentially self-adjoint. \square

We need the following lemma when establishing that the spectrum is discrete.

Lemma 2.1. *Assume there exist a sequence $\{x_n\}_{n \geq 1}$ such that for some $c > 1$, $x_n \leq c^n$, $n \geq 1$ and a sequence $\{h_n\}_{n \geq 1}$ such that $\sum_{n=1}^{\infty} e^{-\sqrt{h_n}\delta_n} < \infty$, where x_n, h_n, δ_n satisfy $V(x, \omega) \geq h_n$, $x_n \leq x \leq x_n + \delta_n$, $n \geq 1$. Then for almost all θ_0 , the spectrum of H^{θ_0} with boundary condition $\cos \theta_0 \psi(0) - \sin \theta_0 \psi'(0) = 0$ is a.s. pure point.*

The Lemma follows from general results, see for example [16]. From the assumptions of the Lemma, it follows that for $\lambda \in \mathbf{R}$, $R_\lambda(0, \cdot) \in L^2([0, \infty), dx)$. Together with the existence of a density for the distribution of $\theta_\lambda(1)$ and Kotani, [13], this implies the conclusion of the Lemma. We now apply this Lemma to show

Corollary 2.2. *For a.e. θ_0 , the spectrum for H^{θ_0} in **Model I** is a.s. pure point.*

Proof. Consider the intervals $\Delta_n = [n^2, (n+1)^2]$ which contains $2n+1$ bumps of unit width. Observe that

$$P\left(\max_{k \in \Delta_n} X_k \leq n\right) = \left(1 - \frac{L(n)}{n^{\alpha/2}}\right)^{(2n+1)} \sim e^{-2L(n)n^{1-\alpha/2}}.$$

Thus, by Borel-Cantelli, a.s. in ω there is an $n_0 = n_0(\omega)$ such that for each $n \geq n_0$ there is an $x_n \in \Delta_n$ with $V(x_n) \geq n$. Setting $\delta_n = 1$, $h_n = n$ an application of Lemma 3.1, shows that the spectrum is pure point a.s.. \square

Another essential and classic result is the following due to Kotani, [13].

Theorem 2.4. *If H^{θ_0} is an essentially self-adjoint operator on $[0, \infty)$ with boundary condition as given in (1) and for a.e. $\lambda \in \mathbf{R}$ there exists a solution $\psi_\lambda \in L^2([0, \infty))$ of the equation $H\psi = \lambda\psi$, then for a.e. $\theta_0 \in [0, 2\pi]$ the spectrum of the operator H^{θ_0} is pure point.*

Remark

Kotani's result will provide the localization result for H^{θ_0} for a.e. $\theta_0 \in [0, 2\pi]$ for all of our models. Note that the spectrum of H^{θ_0} is *P a.s.* the full energy axis in the **Models II, IV** and $[0, \infty)$ for **Models I, III** with a possible single isolated negative

eigenvalue. For such kinds of "solid spectra" the theorem of A. Gordon states that there exists G_δ set $\Gamma \subset [0, \pi]$ with $m(\Gamma) = 0$ so that for every $\theta_0 \in \Gamma$, the spectrum of H^{θ_0} is our singular continuous on $(-\infty, \infty)$ in the case of **Models II, IV** and $(0, \infty)$ for **Models I, III**.

We now give another version of Sch'nol's Theorem, [17], which can be used to determine the nature of the spectrum. Denote the spectral measure of H by ρ .

Theorem 2.5. *Assume the potential V satisfies that there exists a sequence of points y_n with $\lim_{n \rightarrow \infty} y_n = \infty$ and numbers $\delta > 0$, $c > 0$ such that for all n , $V(x) \geq -c$, $x \in (y_n - \delta, y_n + \delta)$. Denote the solutions of*

$$\begin{aligned} -y'' + Vy &= \lambda y, \\ y(0) &= 0, \quad y'(0) = 1. \end{aligned} \tag{25}$$

by ψ_λ . Then, $\forall \epsilon > 0, \exists c_\epsilon > 0$ such that

$$|\psi_\lambda(x)| \leq c_\epsilon x^{\frac{1}{2} + \epsilon}, \text{ for } \rho \text{ a.e. } \lambda.$$

Shnoll's Theorem implies, as observed in as yet unpublished work of Gordon and Molchanov, that given any sequence $\{x_n\}$ with $x_n \rightarrow \infty$,

$$|\psi_\lambda(x_n)| \leq c_\epsilon(\lambda) n^{\frac{1}{2} + \epsilon}, \text{ for } \rho \text{ a.e. } \lambda. \tag{26}$$

This is derived from the inequality using the spectral measure ρ ,

$$\rho\{\lambda : |\psi_\lambda(x_n)| \geq n^{1/2 + \epsilon}\} \leq \frac{\int_0^a |\psi_\lambda(x_n)|^2 \rho(d\lambda)}{n^{1/2 + \epsilon}}$$

followed by an application of the Borel-Cantelli Lemma.

This observation can be used to show that with very large gaps between the bumps in **Model III** the spectrum becomes singular absolutely continuous a.s..

Corollary 2.3. *In Model III if $P(X_n > x) \sim \frac{c}{\ln^{1+\delta} x}$, $x \rightarrow \infty$ for some $\delta > 0$, then the spectral measure ρ is purely singular continuous, i.e. is singular with respect to Lebesgue measure with no eigenvalues.*

Proof. By Furstenburg's Theorem, there is a solution ψ_λ^+ for which $\psi_\lambda^+(L_n)$ is growing exponentially, where L_n is defined at (11). Take $x_n = L_n$ in (26), so that by the above consequence, (26), of Shnoll's Theorem, the spectral measure is singular with respect to Lebesgue measure. Also, the weaker inequality $|\psi_\lambda(L_n)| \geq e^{-(\mu(\lambda) + \epsilon)n}$ holds eventually for any solution where $\mu(\lambda)$ is the Lyapunov exponent. Since $P(X_j \geq e^{cj} \text{ i.o.}) = 1$ it follows that

$$\int_0^{L_n} \psi_\lambda(x) dx \geq \sum_{j=1}^n X_j e^{-2\mu(\lambda)j}.$$

Thus ψ_λ is not an eigenfunction and there are no eigenvalues. □

3 Proofs

All results on the integrated density of states (rotation number) are based on the following elementary variational fact.

Lemma 3.1. *Consider the equation $H^{\theta_0}\psi = \lambda\psi$ on $[0, \infty)$ with the boundary condition set in (1). Assume that on some interval $[0, l_n]$ the potential V is piecewise constant on the subintervals $\Delta = [0, l_1)$, $\Delta = [l_1, l_2)$, \dots , $\Delta_n = [l_{n-1}, l_n]$ with constant value V_i on Δ_i , $i = 1, 2, \dots, n$. Then*

$$\frac{1}{\pi}\theta_\lambda(l_n) = \sum_{k: V_k < \lambda} \sqrt{\lambda - V_k}(l_k - l_{k-1}) + O(n). \quad (27)$$

Corollary 3.1. • *For Model II for any real λ*

$$\frac{1}{\pi}\theta_\lambda(n) = \sum_{k=1}^n \sqrt{|X_k|} + O(n). \quad (28)$$

• *For Model III for any real λ*

$$\frac{1}{\pi}\theta_\lambda(L_n) = \sqrt{\lambda} \sum_{k=1}^n \sqrt{Y_k} + O(n). \quad (29)$$

We first turn attention to **Model I** and obtain a result on the asymptotic distribution of the Markov chain $\{\tilde{t}_{\mathbf{k}}(n) : n \in \{0, 1, 2, \dots\}\}$ for $\theta_{\mathbf{k}}$ the solution of (4), where $\tilde{t}_{\mathbf{k}}(n) = t_{\mathbf{k}}(n) \bmod \pi$. Define $t_{\mathbf{k}}(n) = \tan \theta_{\mathbf{k}}(n)$ and $Z_n = \sqrt{X_n - \mathbf{k}^2} 1_{X_n > \mathbf{k}^2} + \sqrt{\mathbf{k}^2 - X_n} 1_{X_n < \mathbf{k}^2}$. Using (15) we obtain

$$t_{\mathbf{k}}(n+1) = \frac{t_{\mathbf{k}}(n) + \frac{\mathbf{k}}{Z_n} \tanh Z_n}{t_{\mathbf{k}} \frac{Z_n}{\mathbf{k}} \tanh Z_n + 1}, \text{ if } X_n > \mathbf{k}^2, \quad (30)$$

while

$$t_{\mathbf{k}}(n+1) = \frac{t_{\mathbf{k}}(n) + \frac{\mathbf{k}}{Z_n} \tan Z_n}{-t_{\mathbf{k}}(n) \frac{Z_n}{\mathbf{k}} \tan Z_n + 1}, \text{ if } X_n < \mathbf{k}^2. \quad (31)$$

Define function

$$F(t, y) = \frac{t + \frac{\mathbf{k}}{y} \tanh y}{t \frac{y}{\mathbf{k}} \tanh y + 1}, \quad y \geq 0, \quad (32)$$

and

$$G(t, y) = \frac{t + \frac{\mathbf{k}}{y} \tan y}{-t \frac{y}{\mathbf{k}} \tan y + 1}, \quad 0 \leq y \leq \mathbf{k}. \quad (33)$$

Then we have

$$t_{\mathbf{k}}(n+1) = F(t_{\mathbf{k}}(n), Z_n)1_{X_n > \mathbf{k}} + G(t_{\mathbf{k}}(n), Z_n)1_{X_n < \mathbf{k}},$$

from which it is clear that $t_{\mathbf{k}}$ forms a Markov chain. Moreover, there is a one-to-one correspondence between $t_{\mathbf{k}}$ and $\tilde{\theta}_{\mathbf{k}} = \theta_{\mathbf{k}} \bmod \pi$ via the tangent mapping. We will use this to prove the process $\tilde{\theta}_{\mathbf{k}}$ is positive recurrent and has a nice stationary probability distribution and the rate of convergence to this distribution is exponential.

Theorem 3.1. *The Markov chain $\{t_{\mathbf{k}}(n) : n \geq 0\}$ is a Harris m -recurrent chain where m is Lebesgue measure on $(-\infty, \infty)$. The chain possesses a stationary probability measure $\nu_{\mathbf{k}}$ and there is a constant $\gamma \in (0, 1)$ and $C > 0$ such that for any $t \in (-\infty, \infty)$,*

$$\|P(t_{\mathbf{k}}(n) \in \cdot | t_{\mathbf{k}}(0) = t) - \nu_{\mathbf{k}}(\cdot)\|_{TV} \leq C\gamma^n, \quad (34)$$

where $\|\cdot\|_{TV}$ denotes the total variation norm. The measure $\nu_{\mathbf{k}}$ is absolutely continuous with respect to Lebesgue measure and its density is bounded.

Since $\tilde{\theta}_{\mathbf{k}}(n) = \tan^{-1} t_{\mathbf{k}}(n) + \frac{\pi}{2}$, we have the following immediate consequence.

Corollary 3.2. *The Markov chain $\{\tilde{\theta}_{\mathbf{k}}(n) : n \geq 0\}$ is a Harris m -recurrent chain where m is Lebesgue measure on $[0, \pi]$. The chain possesses a stationary probability measure $\mu_{\mathbf{k}}$ and there is a constant $\eta \in (0, 1)$ and $C > 0$ such that for any $\theta \in [0, \pi]$,*

$$\|P(\tilde{\theta}_{\mathbf{k}}(n) \in \cdot | \tilde{\theta}_{\mathbf{k}}(0) = \theta) - \mu_{\mathbf{k}}(\cdot)\|_{TV} \leq C\eta^n. \quad (35)$$

The measure $\mu_{\mathbf{k}}$ is absolutely continuous with respect to Lebesgue measure and its density is bounded.

This corollary implies the existence of the integrated density of states.,

$$N(\mathbf{k}^2) = \lim_{n \rightarrow \infty} \frac{\theta_{\mathbf{k}}(n)}{\pi n}, \quad P - a.s..$$

This follows since

Before we proceed with the proof of Theorem 3.1, we recall a few facts about Harris recurrence from the paper B1, [3]. The following conditions imply the existence of a stationary probability measure for $\{t_{\mathbf{k}}(n) : n \geq 0\}$. Denote the Borel sets in \mathbf{R} by \mathcal{B} .

Condition A

There exists a $C \in \mathcal{B}$ and a probability measure ν and a constant $\beta > 0$ such that

$$P(t_{\mathbf{k}}(1) \in A | t_{\mathbf{k}}(0) = t) \geq \beta \nu(A), \quad \forall t \in C \text{ and } A \in \mathcal{B}. \quad (36)$$

Condition B

There exists a measurable function $W : \mathbf{R} \rightarrow [1, \infty)$ and constants $\lambda < 1$ and $K < \infty$ such that

$$PW(t) \leq \lambda W(t)1_{C^c}(t) + K1_C(t). \quad (37)$$

Condition C

There exists a $\gamma > 0$ such that $\beta\nu(C) \geq \gamma$.

Once we establish Conditions A, B and C for **Model I** we'll have from [3] that Theorem (3.1) holds.

Proof. (Theorem 3.1)

The following condition will imply **Condition A**, and we refer to [1] page 151 for the proof.

Condition D

Suppose the set $C \subset \mathbf{R}$ is such that the density $p(t, s)$ for $\{t_{\mathbf{k}}(n) : n \geq 0\}$ satisfies for some $d_{\mathbf{k}} > 0$

$$p(t, s) \geq d_{\mathbf{k}}, \quad t, s \in C$$

and that C is a recurrent set. Then **Condition A** holds with $\nu(B) = m(B \cap C)/m(C)$ where m is Lebesgue measure.

Notice that **Condition C** holds with this choice of ν since $\nu(C) = 1$.

Lemma 3.2. *Condition D holds*

Proof. For Condition D, set $C = [\frac{1}{\sqrt{3}}, 1]$. Then $p(t, s) \geq d_{\mathbf{k}}, \quad t, s \in C$ will be established using the function F and G defined by (32) and (33). Now for **Model I**, the Ricatti equation (4) is

$$\begin{aligned} \theta_{\mathbf{k}}(n) &= \mathbf{k}n - \frac{1}{\mathbf{k}} \sum_{l=1}^n X_l \int_{l-1}^l \sin^2 \theta_{\mathbf{k}}(z) dz \\ &= \theta_{\mathbf{k}}(n-1) + \mathbf{k} - \frac{X_n}{\mathbf{k}} \int_{n-1}^n \sin^2 \theta_{\mathbf{k}}(z) dz. \end{aligned} \quad (38)$$

Thus, when $X_n < \mathbf{k}^2$, the $\theta_{\mathbf{k}}$ process increases, $\theta_{\mathbf{k}}(n+1) > \theta_{\mathbf{k}}(n)$, while $\theta_{\mathbf{k}}(n+1) < \theta_{\mathbf{k}}(n)$ only when $X_n > \mathbf{k}^2$. This means for the purpose of obtaining the lower bound on $p(t, s)$, we should use G for $t < s$ and F when $t > s$.

So we consider first, $t, s \in A$ with $t < s$. We need to examine $y \rightarrow G(t, y)$. There exists a $y_1 \in [\pi, \frac{3\pi}{2}]$ such that $G(t, y_1) = 1$. For this claim notice that $G(t, \pi) = t < 1$

and $y \rightarrow G(t, y)$ is increasing on $[\pi, \frac{3\pi}{2})$ with $\lim_{y \nearrow \frac{3\pi}{2}} G(t, y) = \infty$ and this gives the existence of y_1 . Since the function $y \rightarrow G(t, y)$ is increasing on $[\pi, y_1]$, when restricted to this interval its inverse exists, and will be denoted by $G^{-1}(t, \cdot)$. Of course, $G^{-1}(t, \cdot) : [t, 1] \rightarrow [\pi, y_1]$. Moreover, we have the derivative in y of G is given by

$$G_y(t, y) = \frac{(\frac{\mathbf{k} \tan y}{y})_y (-\frac{y \tan y}{\mathbf{k}} t + 1) + (t + \frac{\mathbf{k} \tan y}{y}) (\frac{y \tan y}{\mathbf{k}})_y t}{(-\frac{y \tan y}{\mathbf{k}} t + 1)^2} \quad (39)$$

and since $y_1 \in (\frac{\pi}{2}, \frac{3\pi}{2})$, it's easy to see that there is an $M_{\mathbf{k}} < \infty$ such that

$$\sup_{y \in [\frac{\pi}{2}, y_1]} |G_y(t, y)| < M_{\mathbf{k}}. \quad (40)$$

With this in mind, notice that

$$P(t_{\mathbf{k}}(1) \leq s | t_{\mathbf{k}}(0) = t) = P(t_{\mathbf{k}}(1) \leq s, X_1 < \mathbf{k}^2 | t_{\mathbf{k}}(0) = t) + P(t_{\mathbf{k}}(1) \leq s, X_1 < \mathbf{k}^2 | t_{\mathbf{k}}(0) = t)$$

and since both terms on the right hand side are increasing we get

$$p(t, s) \geq \frac{d}{ds} P(t_{\mathbf{k}}(1) \leq s, X_1 < \mathbf{k}^2 | t_{\mathbf{k}}(0) = t).$$

But,

$$\begin{aligned} P(t_{\mathbf{k}}(1) \leq s, X_1 < \mathbf{k}^2 | t_{\mathbf{k}}(0) = t) &= P(G(t, Z_1) \leq s, X_1 < \mathbf{k}^2) \\ &= P(G(t, Z_1) \leq s, Z_1 \in [t, y_1], X_1 < \mathbf{k}^2) \\ &+ P(G(t, Z_1) \leq s, Z_1 \notin [t, y_1], X_1 < \mathbf{k}^2) \\ &= P(Z_1 \leq G^{-1}(t, s), Z_1 \in [t, y_1], X_1 < \mathbf{k}^2) \\ &+ P(G(t, Z_1) \leq s, Z_1 \notin [t, y_1], X_1 < \mathbf{k}^2) \\ &= P(\mathbf{k}^2 - X_1 \leq G^{-1}(t, s)^2, X_1 < \mathbf{k}^2) \\ &+ P(G(t, Z_1) \leq s, Z_1 \notin [t, y_1], X_1 < \mathbf{k}^2) \\ &= \int_{\mathbf{k}^2 - G^{-1}(t, s)^2}^{\mathbf{k}^2} p(u) du \\ &+ P(G(t, Z_1) \leq s, Z_1 \notin [t, y_1], X_1 < \mathbf{k}^2). \end{aligned}$$

Thus,

$$p(t, s) \geq 2p(G^{-1}(t, s)^2) \frac{|G^{-1}(t, s)|}{|G_y(t, G^{-1}(t, s)^2)|}. \quad (41)$$

Using (40) and (41) and the assumption that p is strictly positive it follows that there is a $d_{\mathbf{k}} > 0$ such that

$$\inf_{t, s \in A, t > s} p(t, s) \geq d_{\mathbf{k}}. \quad (42)$$

We now establish the other half of Condition D,

$$\inf_{t,s \in A, t < s} p(t, s) \geq d_{\mathbf{k}}. \quad (43)$$

For this we resort to use of the function F from (32). Simple observation confirms that $F(t, 0) = t + \mathbf{k}$, $\lim_{y \rightarrow \infty} F(t, y) = 0$ and for t fixed, $y \rightarrow F(t, y)$ is strictly decreasing. Denote its inverse in y by $F^{-1}(t, \cdot)$. Therefore, for $t \in A$, there exist $y_1(t) < y_2(t)$ such that $F(t, y_1(t)) = 1$ and $F(t, y_2(t)) = \frac{1}{\sqrt{3}}$. In addition, for y fixed, $t \rightarrow F(t, y)$ is increasing which implies

$$0 < y_1(1) < y_1(t) < y_2(t) < y_2\left(\frac{1}{\sqrt{3}}\right) < \infty, \quad (44)$$

for all $t \in [\frac{1}{\sqrt{3}}, 1]$.

To estimate the density for $t > s$ then, we have

$$\begin{aligned} P(t_{\mathbf{k}}(1) \leq s | t_{\mathbf{k}}(0) = t) &= P(t_{\mathbf{k}}(1) \leq s, X_1 > \mathbf{k}^2 | t_{\mathbf{k}}(0) = t) \\ &+ P(t_{\mathbf{k}}(1) \leq s, X_1 < \mathbf{k}^2 | t_{\mathbf{k}}(0) = t) \end{aligned}$$

and since both terms on the right hand side are increasing in s we conclude that

$$\frac{d}{ds} P(t_{\mathbf{k}}(1) \leq s | t_{\mathbf{k}}(0) = t) \geq \frac{d}{ds} P(t_{\mathbf{k}}(1) \leq s, X_1 > \mathbf{k}^2 | t_{\mathbf{k}}(0) = t)$$

The probability we're differentiating on the right hand side can be simplified,

$$\begin{aligned} P(t_{\mathbf{k}}(1) \leq s, X_1 > \mathbf{k}^2 | t_{\mathbf{k}}(0) = t) &= P(F(t, Z_1) \leq s, X_1 > \mathbf{k}^2) \\ &= P(F(t, Z_1) \leq s, X_1 > \mathbf{k}^2) \\ &= P(Z_1 \leq F^{-1}(t, s), X_1 > \mathbf{k}^2) \\ &= P(X_1 - \mathbf{k}^2 \leq F^{-1}(t, s)^2, X_1 > \mathbf{k}^2) \\ &= P(\mathbf{k}^2 < X_1 \leq F^{-1}(t, s)^2 + \mathbf{k}^2) \\ &= \int_{\mathbf{k}^2}^{F^{-1}(t, s)^2 + \mathbf{k}^2} p(x) dx. \end{aligned}$$

This we have the lower bound for $t, s \in A$ with $t > s$,

$$p(t, s) \geq p(F^{-1}(t, s)^2 + \mathbf{k}^2) \frac{2F^{-1}(t, s)}{F_y(t, F^{-1}(t, s))}. \quad (45)$$

By (44), the range of $F^{-1}(t, s)$ is contained in $[y_1(1), y_2(\frac{1}{\sqrt{3}})] \subset (0, \infty)$ which implies there is a $d_{\mathbf{k}} > 0$ such that

$$\inf_{t,s \in A, t > s} p(F^{-1}(t, s)^2 + \mathbf{k}^2) \frac{2F^{-1}(t, s)}{F_y(t, F^{-1}(t, s))} \geq d_{\mathbf{k}}. \quad (46)$$

□

We now turn to **Condition B**.

Lemma 3.3. *Condition B holds*

Proof. Here we take $W(t) = 1_{C^c}(t)$. Then $PW(t) = P(t_{\mathbf{k}}(1) \in C^c | t_{\mathbf{k}}(0) = t)$. Note that $PW(t) \leq 1$ for all t so we can take $K = 1$. The other requirement splits into two cases. In the case $t < \frac{\sqrt{3}}{3}$ we have for $Z_1 = \sqrt{\mathbf{k}^2 - X_1} 1_{\{\mathbf{k}^2 > X_1\}}$. Since X_1 has a nonvanishing density p we have for all $t < \frac{\sqrt{3}}{3}$,

$$P(t_{\mathbf{k}}(1) \in C | t_{\mathbf{k}}(0) = t) \geq \underset{>}{P(G(t, Z_1) \in [\frac{\sqrt{3}}{3}, 1]; \mathbf{k}^2 > X_1)} \underset{0}{}$$

and again using the fact that X_1 has a nonvanishing density p

$$\lim_{t \rightarrow -\infty} P(G(t, Z_1) \in [\frac{\sqrt{3}}{3}, 1]; \mathbf{k}^2 > X_1) = \underset{>}{P(\frac{\frac{-1}{\sqrt{\mathbf{k}^2 - X_1}}}{\frac{1}{\mathbf{k}} \tan \sqrt{\mathbf{k}^2 - X_1}} \in [\frac{\sqrt{3}}{3}, 1]; \mathbf{k}^2 > X_1)} \underset{0}{}$$

Thus, there is an $\epsilon > 0$ such that for all $t < \frac{\sqrt{3}}{3}$,

$$P(t_{\mathbf{k}}(1) \in C | t_{\mathbf{k}}(0) = t) > \epsilon. \tag{47}$$

□

In the second case where $t > 1$, taking $Z_1 = \sqrt{X_1 - \mathbf{k}^2} 1_{\{\mathbf{k}^2 < X_1\}}$

$$P(t_{\mathbf{k}}(1) \in C | t_{\mathbf{k}}(0) = t) \geq \underset{>}{P(F(t, Z_1) \in [\frac{\sqrt{3}}{3}, 1]; \mathbf{k}^2 > X_1)} \underset{0}{}$$

Using the fact that X_1 has a nonvanishing density p

$$\lim_{t \rightarrow -\infty} P(F(t, Z_1) \in \left[\frac{\sqrt{3}}{3}, 1\right]; \mathbf{k}^2 > X_1) = \underset{>}{P(\frac{1}{\frac{\sqrt{X_1 - \mathbf{k}^2}}{\mathbf{k}} \tan \sqrt{X_1 - \mathbf{k}^2}} \in [\frac{\sqrt{3}}{3}, 1]; \mathbf{k}^2 > X_1)} \underset{0}{}$$

Thus, taking another value of ϵ if necessary one has

$$P(t_{\mathbf{k}}(1) \in C | t_{\mathbf{k}}(0) = t) \geq \epsilon.$$

Thus,

$$PW(t) \leq (1 - \epsilon)W(t), t \notin C$$

and Condition B is proved. □

Proof. (Theorem 1.1) We first establish the existence of the integrated density of states.

The limiting joint distribution of the Markov chain $(\mathcal{S}_{\mathbf{k}}(n) := (X_{n+1}, \tilde{\theta}_{\mathbf{k}}(n)))$ is

$$p(y)\mu_{\mathbf{k}}(d\psi) dy$$

since the components X_{n+1} and $\tilde{\theta}_{\mathbf{k}}(n)$ are independent.

This implies that the Markov sequence of processes

$$(X_{n+1}, \{\tilde{\theta}_{\mathbf{k}}(n+t) : 0 \leq t \leq 1\}_{n=0}^{\infty})$$

has a limiting distribution on the space $[0, \infty) \times C([0, 1], [0, \pi])$.

The Ricatti equation (4),

$$\theta_{\mathbf{k}}(n) = \theta_{\mathbf{k}}(n-1) + \mathbf{k} - \frac{X_n}{\mathbf{k}} \int_{n-1}^n \sin^2 \theta_{\mathbf{k}}(z) dz. \quad (48)$$

implies

$$\theta_{\mathbf{k}}(n) - \theta_{\mathbf{k}}(n-1) = \mathbf{k} - H(X_{n+1}, \tilde{\theta}_{\mathbf{k}}(n))$$

where, due to smooth dependence on initial date, H is a bounded smooth function on $[0, \infty) \times [0, \pi]$.

Therefore, by the ergodic theorem for Markov chains, we derive the existence of the integrated density of states,

$$\begin{aligned} N(\mathbf{k}^2) &= \frac{1}{\pi} \lim_{n \rightarrow \infty} \frac{\theta_{\mathbf{k}}(n)}{n} \\ &= \frac{1}{n} \sum_{j=0}^{n-1} (\theta_{\mathbf{k}}(j) - \theta_{\mathbf{k}}(j-1)) \\ &= \mathbf{k} - \frac{1}{n} \sum_{j=0}^{n-1} H(X_{n+1}, \tilde{\theta}_{\mathbf{k}}(n)) \\ &= \mathbf{k} - \int_0^\pi \int_0^\infty H(x, \psi) p(x) \mu_{\mathbf{k}}(\psi) dx. \end{aligned} \quad (49)$$

It follows easily that,

$$\lim_{\mathbf{k} \rightarrow \infty} \frac{N(\mathbf{k}^2)}{\mathbf{k}} = \frac{1}{\pi}.$$

Next we establish the existence of the random Lyapunov exponent in the nonlinear scale $n^{1/\alpha}$. First we remark that the stationary distribution $\mu_{\mathbf{k}}$ has a bounded density. This follows from the smoothness of H above and the fact that the density p is bounded. These imply that the transition density $p(s, t)$ is bounded. Thus, for measurable $A \subset [0, \pi]$,

$$\begin{aligned} \mu_{\mathbf{k}}(A) &= \int_0^\pi \mu_{\mathbf{k}}(d\psi) p(\psi, A) \\ &\leq \|p(\cdot, \cdot)\|_\infty |A|. \end{aligned}$$

Use $\mathcal{A}_{\mathbf{k}}(x)$ as above to denote the one-step monodromy matrix defined at (15) and (18) for the cases $x \leq \mathbf{k}^2$ and $x > \mathbf{k}^2$, respectively. As shown above, the process $\{\tilde{\theta}_{\mathbf{k}}(n) : n \geq 0\}$ and $\{\mathcal{S}_{\mathbf{k}}(n) := (X_{n+1}, \tilde{\theta}_{\mathbf{k}}(n)) : n \geq 0\}$ are Markov chains with nice stationary distributions $\mu_{\mathbf{k}}(d\theta)$ and $\pi_{\mathbf{k}}(dx, d\theta) = p(x)\mu_{\mathbf{k}}(d\theta)dx$, respectively. This allows us to apply Furstenberg's Theorem in the argument below.

For $\mathbf{v} = (v_1, v_2) \neq \mathbf{0}$, write $\theta_{\mathbf{k}}(0) = \frac{v}{\|\mathbf{v}\|}$ and $\theta_{\mathbf{k}}(j) = \frac{\mathcal{M}_{\mathbf{k}}([0, j]\mathbf{v})}{\|\mathcal{M}_{\mathbf{k}}([0, j]\mathbf{v})\|}$ where we identify these quantities by means of (2) with $\mathcal{M}_{\mathbf{k}}([0, j]\mathbf{v})$ replacing $(\psi_{\mathbf{k}}(j), \psi'_{\mathbf{k}}(j))$ so that

$$\begin{aligned} \ln \frac{\|\mathcal{M}_{\mathbf{k}}([0, n])\mathbf{v}\|}{\|\mathbf{v}\|} &= \ln \frac{\|\mathcal{A}_{\mathbf{k}}(X_n)\mathcal{M}_{\mathbf{k}}([0, n-1])\mathbf{v}\|}{\|\mathcal{M}_{\mathbf{k}}([0, n-1])\mathbf{v}\|} + \ln \frac{\|\mathcal{A}_{\mathbf{k}}(X_{n-1})\mathcal{M}_{\mathbf{k}}([0, n-2])\mathbf{v}\|}{\|\mathcal{M}_{\mathbf{k}}([0, n-2])\mathbf{v}\|} \\ &\quad + \dots + \ln \frac{\|\mathcal{A}_{\mathbf{k}}(X_1)\mathbf{v}\|}{\|\mathbf{v}\|} \\ &= \sum_{j=0}^{n-1} \ln \|\mathcal{A}_{\mathbf{k}}(X_{j+1})\theta_{\mathbf{k}}(j)\| \\ &= \sum_{j=0, X_j \geq \mathbf{k}^2}^{n-1} \ln \|\mathcal{A}_{\mathbf{k}}(X_{j+1})\theta_{\mathbf{k}}(j)\| + \sum_{j=0, X_j < \mathbf{k}^2}^{n-1} \ln \|\mathcal{A}_{\mathbf{k}}(X_{j+1})\theta_{\mathbf{k}}(j)\| \\ &= I_n + II_n. \end{aligned}$$

so $\ln \|\mathcal{M}_{\mathbf{k}}([0, n])\theta\|$ is an additive functional of the Markov chain $\{\mathcal{S}_{\mathbf{k}}(n) : n \geq 0\}$.

$$\ln \frac{\|\mathcal{M}_{\mathbf{k}}([0, n])\mathbf{v}\|}{\|\mathbf{v}\|} = \sum_{j=0}^{n-1} f(\mathcal{S}_{\mathbf{k}}(j))$$

with

$$f(x, \mathbf{w}) = \ln \frac{\|A(x)\mathbf{w}\|}{\|\mathbf{w}\|}.$$

By Furstenberg's Theorem,

$$\lim_{n \rightarrow \infty} \frac{1}{n} II_n = \eta >, \text{ a.s.}$$

and consequently,

$$\lim_{n \rightarrow \infty} \frac{1}{n^{1/\alpha}} II_n = 0, \text{ a.s.}$$

By our assumption on the density of the iid sequence $\{X_j : j \geq 1\}$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{j \leq n : X_j > \mathbf{k}^2\}| = \int_{\mathbf{k}^2}^{\infty} p(x)dx > 0.$$

Thus we only need show that $\frac{1}{n^{1/\alpha}} I_n$ has a limiting St_{α} distribution. According to a general result in [12] about limits of additive functionals, the claim of the Theorem follows if we establish that

$$\int_{f(x, \theta) > z} \pi_{\mathbf{k}}(dx, d\theta) \sim c_1 z^{-2\alpha} \quad \text{as } z \rightarrow \infty$$

which translates to

$$\int_{||A(x)\theta|| > e^z ||\theta||} p(x) \mu_{\mathbf{k}}(d\theta) dx \sim c_1 z^{-2\alpha} \quad \text{as } z \rightarrow \infty.$$

If $x > \mathbf{k}^2$, the matrix $\mathcal{A}_{\mathbf{k}}(x)$ has two positive eigenvalues

$$t_{\pm}(x) = e^{\pm \sqrt{x - \mathbf{k}^2}}$$

with corresponding eigenvectors

$$v_{\pm} = \left(1, \pm \frac{y}{\mathbf{k}}\right)^T.$$

so $\ln t_+(X_n) = \sqrt{X_n - \mathbf{k}^2}$ belongs to the domain of attraction of the stable law of index α . Also, the matrix $\mathcal{D}_{\mathbf{k}}(x) := (\mathcal{A}_{\mathbf{k}}(x)^* \mathcal{A}_{\mathbf{k}}(x))^{1/2}$ has eigenvalues

$$\mu_{\pm}(x) = \frac{\sqrt{b+2} \pm \sqrt{b-2}}{2}$$

with $b = 2 \cosh^2 a + (a^2 + 1/a^2) \sinh^2 a$ and $a = \sqrt{x - \mathbf{k}^2}$. So $\mu_+(x)$ can be written as

$$\mu_+(x) = \left(a + \frac{1}{a}\right) |\sinh a| \frac{1 + \sqrt{1 + \frac{4}{(a + \frac{1}{a})^4 \sinh^2 a}}}{2}$$

and since

$$\frac{\ln \mu_+(x)}{a} \rightarrow 1 \quad \text{as } a \rightarrow \infty$$

it follows that $\log \mu_+(X_n)$ also belongs to the domain of attraction of the stable law of index α .

Now

$$\int_{||A(x)\theta|| > e^z ||\theta||} p(x) \mu_{\mathbf{k}}(d\theta) dx = \int_{\mu_+(x) > z} \mu_{\mathbf{k}}(E_x) p(x) dx$$

with $E_x := \{\theta : ||\mathcal{D}_{\mathbf{k}}(x)\theta|| > e^z ||\theta||\}$. Since $\mathcal{D}_{\mathbf{k}}(x)$ is selfadjoint, the set E_x is in fact a cone centered at the eigenvector of μ_+ . Its Lebesgue measure can be computed explicitly and is

$$g(x, z) = 2 \arctan \sqrt{\frac{e^{2(\mu_+(x) - z)} - 1}{1 - e^{-2(\mu_+(x) + z)}}}.$$

Since $\mu_{\mathbf{k}}(d\theta)$ has a density $m_{\mathbf{k}}(\theta)$ w.r.t. Lebesgue measure, satisfying $m \leq m_{\mathbf{k}}(\theta) \leq M$, we have

$$m \int_{\mu_+(x) > z} g(x, z) p(x) dx \leq \int_{\mu_+(x) > z} \mu_{\mathbf{k}}(E_x) p(x) dx \leq M \int_{\mu_+(x) > z} g(x, z) p(x) dx$$

therefore the proof is completed with the application of the following lemma, with

$$h(u, z) = 2 \arctan \sqrt{\frac{e^{2(u-z)} - 1}{1 - e^{-2(u+z)}}}$$

and $U = \mu_+(X_1)$. □

Lemma 3.4. *If $U > 0$ is a r.v. in the domain of attraction of the stable law of index $0 < \alpha < 2$, and $h(u, z)$ is a function satisfying*

$$\begin{aligned} \lim_{u \rightarrow +\infty} h(u, z) &= k, \quad \forall z \\ \lim_{z \rightarrow +\infty} h(tz, z) &= k, \quad \forall t > 1 \end{aligned} \tag{50}$$

then there is a $k' > 0$ such that

$$\mathbb{E}[h(U, z); U > z] \sim k' z^{-\alpha} \quad \text{as } z \rightarrow +\infty.$$

Proof. Let F be the distribution function of U , so that $z^\alpha(1 - F(z)) = c(1 + o(1))$ as $z \rightarrow \infty$. Then

$$\begin{aligned} z^\alpha \mathbb{E}[h(U, z); U > z] &= z^\alpha \int_{u>z} h(u, z) dF(u) \\ &= z^\alpha \int_{t>1} h(tz, z) dF(zt), \quad t = \frac{u}{z}. \end{aligned}$$

By (50) we have $\forall \epsilon > 0$

$$z^\alpha \int_{t>1} (k - \epsilon) dF(zt) < z^\alpha \int_{t>1} h(tz, z) dF(zt) < z^\alpha \int_{t>1} (k + \epsilon) dF(zt)$$

for z large enough, which going back to the variable u becomes

$$z^\alpha \int_{u>z} (k - \epsilon) dF(u) < z^\alpha \int_{u>z} h(u, z) dF(u) < z^\alpha \int_{u>z} (k + \epsilon) dF(u).$$

Hence the statement of the Lemma with $k' = kc$. □

Proof. We now prove Theorem (1.2) for **Model II**. In view of (17), for $n \leq x \leq n+1$, we can write when $X_l + \lambda \geq 0$.

$$\begin{aligned} \begin{pmatrix} \psi_\lambda(x) \\ \frac{1}{\lambda} \psi'_\lambda(x) \end{pmatrix} &= \begin{pmatrix} \cos(\sqrt{\lambda + X_n}(x - n)) & \frac{|\lambda|}{\sqrt{\lambda + X_n}} \sin(\sqrt{\lambda + X_n}(x - n)) \\ -\frac{\sqrt{\lambda + X_n}}{\lambda} \sin(\sqrt{\lambda + X_n}(x - n)) & \cos(\sqrt{\lambda + X_n}(x - n)) \end{pmatrix} \\ &\quad \times \begin{pmatrix} \psi_\lambda(n) \\ \frac{1}{\lambda} \psi'_\lambda(n) \end{pmatrix} \end{aligned} \tag{51}$$

from which it's clear that the vector

$$\begin{pmatrix} \psi_\lambda(x) \\ \frac{1}{\lambda}\psi'_\lambda(x) \end{pmatrix}$$

traces out a curve through a total angle of $\sqrt{\lambda + X_l}$. That is

$$\theta_\lambda(n+1) = \theta_\lambda(n) + \sqrt{\lambda + X_n}.$$

Thus,

$$\theta_\lambda(n) = \theta_\lambda(0) + \sum_{l=1}^n \sqrt{\lambda + X_l} \quad (52)$$

Due to the assumption on the distribution of the $\{X_l\}$ it follows that

$$\lim_{n \rightarrow \infty} \frac{\theta_\lambda(n)}{\pi n^{1/\alpha}} \stackrel{\mathcal{L}}{=} \zeta_\alpha \quad (53)$$

with ζ_α a random variable with an St_α distribution. This implies

$$\lim_{n \rightarrow \infty} \frac{\theta_\lambda(n)}{\pi n} = \infty, \text{ a.s..} \quad (54)$$

From (52) it follows that $\theta_\lambda(n) \bmod \pi$ has a nice density. Also, by the assumption on the distribution of X_n it follows that $E[\ln^+ \|\mathcal{M}_\lambda(n)\|] < \infty$ and so by standard results, $\gamma(\lambda)$ exists *P a.s.* and is positive. From Kotani's results, [13], we can then conclude the spectrum is pure point *a.s.* for *a.e.* θ_0 since the resolvent in this case is in L^2 . The growth rate for the eigenfunctions also follows by Kotani's method. \square

Proof. We now prove Theorem (1.3) for (**Model III**). Since the random variables Y_l possess a density, the Ricatti equation (4) implies that the distribution of the phase $\theta_{\mathbf{k}}$ has a density (note that the distribution of $\mathbf{k}Y_l \bmod \pi$ has a bounded density) and since $\mathcal{M}_{\mathbf{k}}([0, L_n]) = \mathcal{C}_{\mathbf{k}}(Y_n) \cdots \mathcal{C}_{\mathbf{k}}(Y_2)\mathcal{C}_{\mathbf{k}}(Y_1)$, with $E[\ln^+ \|\mathcal{C}_{\mathbf{k}}(Y_1)\|] < \infty$, the Furstenberg Theorem holds:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln \|\mathcal{M}_{\mathbf{k}}([0, L_n])\| = \gamma(\mathbf{k}^2)_{\mathbf{n}l} > 0, \mathbf{P} - \mathbf{a.s..}$$

The Lyapunov exponent with respect to the linear scale then satisfies

$$\gamma(\mathbf{k}^2) = \lim_{\mathbf{n} \rightarrow \infty} \frac{1}{\mathbf{n}} \ln \|\mathcal{M}_{\mathbf{k}}([0, \mathbf{n}])\| = 0, \mathbf{a.s..} \quad (55)$$

This follows easily from the fact that $L(n) \sim n^{1/\alpha}$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \ln \|\mathcal{M}_{\mathbf{k}}([0, n])\| &= \lim_{n \rightarrow \infty} \frac{\ln \|\mathcal{M}_{\mathbf{k}}([0, n])\|}{\ln \|\mathcal{M}_{\mathbf{k}}([0, L_n])\|} \frac{\ln \|\mathcal{M}_{\mathbf{k}}([0, L_n])\|}{n} \\ &= \gamma(\mathbf{k}^2)_{\mathbf{n}l} \lim_{\mathbf{n} \rightarrow \infty} \frac{\ln \|\mathcal{M}_{\mathbf{k}}([0, \mathbf{n}])\|}{\ln \|\mathcal{M}_{\mathbf{k}}([0, \mathbf{L}_{\mathbf{n}}])\|} \\ &= 0. \end{aligned}$$

Finally, observe that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\ln ||\mathcal{M}_{\mathbf{k}}([0, L_n])||}{L_n^\alpha} &= \lim_{n \rightarrow \infty} \frac{\ln ||\mathcal{M}_{\mathbf{k}}([0, L_n])||}{n} \frac{n}{L_n^\alpha} \\ &= \frac{\gamma(\mathbf{k}^2)_{\mathbf{n}!}}{\zeta_\alpha^\alpha}, \end{aligned}$$

where ζ_α again has a St_α distribution.

We consider now the asymptotic behavior of the rotation number $\theta_{\mathbf{k}}(n)$. By the Ricatti equation (4), across the n^{th} gap the rotation is

$$\theta_{\mathbf{k}}(L_n - 1) - \theta_{\mathbf{k}}(L_{n-1}) = \mathbf{k} Y_n,$$

whereas across the n^{th} bump the rotation is

$$\theta_{\mathbf{k}}(L_n) - \theta_{\mathbf{k}}(L_n - 1) = O(1).$$

Thus,

$$\theta_{\mathbf{k}}(L_n) = \mathbf{k} \sum_{j=1}^n Y_j + O(n) \quad (56)$$

which implies by convergence to the stable law St_α

$$\lim_{n \rightarrow \infty} \frac{\theta_{\mathbf{k}}(L_n)}{n^{1/\alpha}} = \mathbf{k} \zeta_\alpha.$$

But also from (56) follows

$$\lim_{n \rightarrow \infty} \frac{\theta_{\mathbf{k}}(L_n)}{L_n} = \mathbf{k}$$

and consequently,

$$N(\mathbf{k}^2) = \frac{\mathbf{k}}{\pi}.$$

We turn now to consideration of the spectrum. Since $\ln ||M_{\mathbf{k}}([0, L_n])|| \sim \gamma(\mathbf{k}^2)n$, and by (4) the magnitude of $r_{\mathbf{k}}(x)$ is constant across the gaps, there are two so-called Weil solutions on $[0, \infty)$, one of which, $\psi_{\mathbf{k}}^+$, has the maximal rate of growth i.e.

$$\psi_{\mathbf{k}}^+(L_n) \leq C e^{(\gamma(\mathbf{k}^2) + \epsilon)n}, \quad n \geq n_0(\omega), \quad P - a.s.. \quad (57)$$

The second solution, $\psi_{\mathbf{k}}^-$, is exponentially decreasing, i.e.

$$\psi_{\mathbf{k}}^-(L_n) \leq C e^{(-\gamma(\mathbf{k}^2) + \epsilon)n}, \quad n \geq n_0(\omega), \quad P - a.s.. \quad (58)$$

Note that by the Ricatti equation (4), the function $r_{\mathbf{k}}$ is constant on the intervals between bumps. The Green function, $R_{\mathbf{k}+i0}(0, L_n)$ in this case, is decreasing exponentially with rate at least $\gamma(\mathbf{k}^2) - \epsilon$. Thus, $P - a.s.$,

$$\int_0^\infty |R_{\mathbf{k}+i0}(0, x)|^2 dx \leq \sum_{n=0}^\infty Y_n e^{-2(\gamma(\mathbf{k}^2) - \epsilon)n} + O(1). \quad (59)$$

The fluctuations of the sequence Y_n can be estimated by noting

$$P(Y_n > n^{1/\alpha} \ln^{2/\alpha} n) \sim \frac{c}{n \ln^2 n},$$

so that by Borel-Cantelli,

$$P(Y_n > n^{1/\alpha} \ln^{2/\alpha} n, i.o.) = 0.$$

This implies by (59) that *a.s.* $R_{\mathbf{k}+i0}(0, \cdot) \in L^2([0, \infty))$. Using this with Kotani's result and the absolutely continuous distribution of the phase $\theta_{\mathbf{k}}(S_n)$, give the localization result, namely, for a.e. θ_0 , the spectrum of H^{θ_0} is pure point *a.s.* \square

Remark 3.1. *We wish to stress again that the fact that the spectrum in Model III is pure point while the Lyapunov exponent is identically zero doesn't contradict the "Kotani Theory." The reason of course is that the point potential in Model III is not ergodic.*

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